## 2021 James S. Rickards Fall Invitational

- 1. The 5 number summary of a sample is the minimum, 1st quartile, median, 3rd quartile, and maximum. Using 1-Var Stats, we get 2, 13, 43, 79, and 646. The sum of these is 783, or C.
- 2. This equation represents a semicircle (above the x-axis) centered at (10, 0) with radius r. The area of this semicircle is  $\frac{1}{2}\pi r^2$ . Setting this equal to 1 we get  $r = \boxed{0.798, \text{ or C.}}$
- 3. To find all possible paths disregarding the line, we find all possible permutations of 6 indistinguishable right moves and 6 indistinguishable up moves. This is  $\frac{12!}{(6!)(6!)} = 924$ . To account for the uncrossable line, we draw out the grid and the line. By summing up all the possible paths at each point above the line using the below diagram, we eventually get 429.



4. The frog can end up 6 units away (6 in the same direction), 4 units away (5 in one direction, 1 in the other), 2 units away (4 in one direction, 2 in the other), or 0 units away (3 in one direction, 3 in the other). The probability that the frog ends up 6 units away is  $2(\frac{1}{2^6}) = \frac{2}{2^6}$  because we must double to account for both directions. The probability that the frog ends up 4 units away is  $2(\frac{1}{2^6})(\frac{6!}{(5!)(1!)}) = \frac{12}{2^6}$ . The probability that the frog ends up 2 units

away 
$$2(\frac{1}{2^6})(\frac{6!}{(4!)(2!)}) = \frac{30}{2^6}$$
. Thus the expected value is  $\frac{0(20)+2(30)+4(12)+6(2)}{2^6} = \frac{120}{64} = \boxed{\frac{15}{4}}$ , or A.

- 5. We can find the expected by solving for x in x + 2x + 3x + 4x + 5x + 6x = 1, since the probabilities are weighted by value, and all probabilities add to 1. We can then find the expected value of the total rolls by multiplying the probability of getting each value by the total 105 roles. Doing this, we get 5, 10, 15, 20, 25, 30. Using  $\chi^2 GOF$ , with 7, 8, 12, 27, 22, 29 for observed, and 5, 10, 15, 20, 25, 30 for expected, we get the p-value as 0.461, or D.
- 6.  $E(X^2) = E(X)^2 + Var(X)$ , and E(X+Y) = E(X) + E(Y), so  $E(X^2) = 5^2 + 4^2 = 41$ .  $E(Y^2) = 3^2 + 3^2 = 18$ .  $E(X^2) + E(Y^2) + X + Y = 41 + 18 + 5 + 3$  equals 67, or B.
- 7.  $\mu_{ax} = a\mu_x$ ,  $\sigma_{ax} = a\sigma_x$ ,  $\mu_{x+y} = \mu_x + \mu_y$ , and  $\sigma_{x+y} = \sqrt{\sigma_x^2 + \sigma_y^2}$ , so the mean weight of 30 tennis balls is 30 \* 1 = 30 lbs, and the standard deviation is 0.5 \* 30 = 15 lbs. Adding that to the mean and standard deviation of the box ( $\mu = 2$ ,  $\sigma = 0.35$ ), we get the mean weight of the whole box is 30 + 2 = 32 lbs, and the standard deviation is  $\sqrt{15^2 + 0.35^2} = 15.004$  lbs. The mean weight of 5 boxes are 5 \* 32 = 160 lbs, with a standard deviation of 15.004 \* 5 = 75.02 lbs. Adding that to the mean and standard deviation of the truck ( $\mu = 150$ ,  $\sigma = 10.25$ ), we get the mean weight of the whole truck is 150 + 160 = 310 lbs, and the standard deviation is  $\sqrt{75.02^2 + 10.25^2} = 75.717$  lbs. Since the truck's weight is normally distributed (adding normally distributed random variables together makes the sum also normally distributed), we can use normalcdf to find the probability that the truck's weight is between 300 and 350, so normalcdf(300, 350, 310, 75.717) = 0.254, or D.]
- 8. Since we are trying to find the probability of the kth success on the nth trial, this distribution describes Negative Binomial, or C.

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- 9. This is a pretty well-known probability question. A good explanation is found here. If we fix two points on the circle, and make the third point variable, we can show that there is an arc on the circle where p3 can exist such that the triangle formed between the three points contains the center of the circle. This arc is found by drawing a line through p1 and the center, and p2 and the center, so the circle is split up into 4 arcs. If p3 is in the arc opposite p1 and p2, then the triangle formed contains the center. The average size of this arc is  $\frac{1}{4}$  of the circumference, because the maximum size the arc can be is  $\frac{1}{2}$  of the circumference, and the smallest size the arc can be is 0, with every size in that range equally likely. So since anywhere in that arc satisfies our condition, the probability of this happening is  $\frac{1}{4}$ , or D.
- 10. The mean is always an unbiased estimator, and the median is an unbiased estimator if the population is normal. All the other statistics are biased estimators of the population parameter. So, the answer is 2, or B.
- 11. This situation describes a binomial distribution, with 4 successes and 6 failures. Using binompdf, we can find the probability of this happening. binompdf(10, 0.65, 4) = 0.069, or E.
- 12. Let X be the event that the original choice was correct, let S be the event that you switch doors, and let W be the event that you win the prize. We are looking for P(W|S) and  $P(W|S^C)$ . By conditioning,  $P(W|S) = P(X)P(W|S,X) + P(X^C)P(W|S,X^C)$ . We know that  $P(X) = \frac{1}{10}$  because you originally choose 1 out of 10 doors. Also, P(W|S,X) = 0 because if you had the right door and switched, you would have no chance of winning the prize. But  $P(W|S,X^C) = \frac{1}{8}$  because if you had the wrong door and switched, you would have a 1 in 8 chance of guessing the right one from the remaining 8 doors. Thus  $P(W|S) = \frac{1}{10}(0) + \frac{9}{10}(\frac{1}{8}) = \frac{9}{80}$ . Similarly,  $P(W|S,X^C) = P(X)P(W|S^C,X) + P(X^C)P(W|S^C,X^C) = \frac{1}{10}(1) + \frac{9}{10}(0) = \frac{1}{10}$  because the only way you can win by not switching is if you had the good fortune of choosing the correct door the first time. So we have  $A/B = \frac{9}{80}/\frac{1}{10} = \frac{9}{8}$ , or B.
- 13. To find the power, we first need to find the upper bound of the region of acceptance, or the largest value such that the *p*-value of the significance test would be below 0.05. The upper bound,  $b = \mu + CV * SE$ , where CV is the critical value corresponding to the alpha value, and the SE is the standard error. The CV of 0.05 is 1.645 (invNorm for 90%), and the standard error is  $\frac{2.54}{\sqrt{100}}$ , or 0.254, so b = 55 + 1.645 \* 0.245 = 55.403. Since we are assuming the null hypothesis is false, and the true population mean is 56, we find the power by calculating the probability of getting a value more extreme than b in the direction of the alternative hypothesis, so P(Z > z) where  $z = \frac{b-\mu}{SE}$ , or  $\frac{55.403-56}{2.54} = -2.35$ . So, the power = p(Z > -2.35). Since this sample is normally distributed according to Central  $\sqrt{100}$ .

Limit Theorem, p(Z > -2.35) = normalcdf(-2.35, 1e99, 0, 1) = 0.991, or A.

- 14. Graphing this sample on a calculator gives a histogram that looks most closely to bimodal, or C.
- 15. Since age would be the explanatory variable, and the number of homeruns would be the response variable, we can use LinReg on your calculator to find the slope and correlation coefficient, which are 1.131 and 0.322 respectively. Adding them together gives 1.453, or C.
- 16. Putting 12 in the LSRL we calculated in question 15, we get 9.413. Since the residual is  $y-\hat{y}$ , the residual = 10–9.413  $\rightarrow 0.587$ , or B.
- 17. This sample does not pass Central Limit Theorem, because the sample size is less than 30. Central Limit Theorem, or C.
- 18. Using TInterval in your calculator, we can find the 95% confidence interval, which has bounds at (8.7416, 16.758). Adding these together, and rounding gives 25, or B.
- 19. We can solve this problem using a tree diagram. There is a probability of 0.05 that a competitor has the disease, and a probability of 0.95 that they don't. If they do have the disease, there is a 0.95 chance that the test gives a true positive, and a 0.05 chance that it gives a false negative. If they don't have the disease, there is a 0.1 chance that the test gives a false positive, and a 0.9 chance that it gives a true negative. Using conditional probability, we are finding out  $P(positive \cup disease) + P(negative \cup nodisease) = (0.05 * 0.95) + (0.95 * 0.90) = 0.903$ , or A.

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- 20. The conditions for a hypergeometric distribution are that each trial is not independent, the probability of success is the same for each trial, each observation is of two outcomes, and the number of trials is fixed. Of the four choices given, only 2 match these conditions, or A.
- 21. Let x be Ananya's arrival time (in minutes after 2:00) and y be Akhil's arrival time (in minutes after 2:00). Akhil must arrive before 20 minutes have passed since Ananya's arrival thus y < x + 20. But Akhil must arrive after 10 minutes before Ananya's arrival thus y > x 10. Also, since each of them must arrive after 2:00 and before 3:00, x > 0, y > 0, x < 60, y < 60. Graphing each of these regions we see they partition a 60 x 60 square into a hexagon and two triangles. The probability of overlap is the area of the hexagon divided by the area of the square. The area of the hexagon is the area of the square (3600) minus by the sum of the areas of the two triangles:  $\frac{1}{2}(40)(40) + \frac{1}{2}(50)(50) = 2050$ . So our final answer is  $\frac{3600-2050}{3600} \approx 0.431$ , or B.
- 22.  $\mu_{X^2} = \mu_X^2 + \sigma_X^2$ , so  $\mu_{X^2} = 0^2 + 3^2 = 9$ .  $Var(X^2) = \mu_{X^4} \mu_{X^2}^2$ , so  $Var(X^2) = 97-81 = 16$ .  $\mu_X + \mu_Y = 9+9 = 18$ , and  $\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2r\sigma_X\sigma_Y}$ , so  $\sigma_{X+Y} = \sqrt{4^2 + 3^2 + 2(0.28)(4)(3)} = 5.632$ . Since X and Y are normal, we can use normalcdf to find the probability of  $X^2 + Y$  being between 15 and 25, so normalcdf(15, 25, 18, 5.632) = 0.5959, or D.
- 23. Since  $P(A \cup B) = P(A) + P(B) P(A \cap B)$ , to minimize or maximize  $P(A \cup B)$ , we need to maximize or minimize  $P(A \cap B)$  respectively. The maximum value of  $P(A \cap B)$  is min(P(A), P(B)), and the minimum value of  $P(A \cap B)$  is 0. So, the minimum value of  $P(A \cup B) = 0.56 + 0.29 0.29 = 0.56$ , and the maximum value of  $P(A \cup B) = 0.56 + 0.29 0.29 = 0.56$ , and the maximum value of  $P(A \cup B) = 0.56 + 0.29 0 = 0.85$ . Adding these together gives 1.41, or C.
- 24. Let P(i) be the probability that he is in Seat A on the *i*-th day. For Mihir to be in Seat A on day *i*, he must have either been in Seat A on day i-1 and satisfied an 80% criterion, or been in Seat B on day i-1 and satisfied a 10% criterion. Thus P(i) = 0.8P(i-1) + 0.1(1 P(i-1)) = 0.7P(i-1) + 0.1. Using this equation, P(i) < P(i-1) as long as  $P(i-1) > \frac{1}{3}$ . So since  $P(1) = 1 > \frac{1}{3}$ , as *i* increases the probability of being in Seat A will continually decrease toward  $\frac{1}{3}$ , and the probability of being in Seat B will continually increase toward  $\frac{2}{3}$ . Though the probability will not be exactly  $\frac{2}{3}$  until an infinite number of days have passed, 180 days is large enough to be certain to one decimal place (this can be realized by plugging the transition into a calculator more than 10 times it converges rapidly). Therefore, the answer is 0.7, or E.
- 25. The pooled standard error of two proportions is  $\sqrt{p0(1-p0)(\frac{1}{n1}+\frac{1}{n2})}$ , where  $p0 = \frac{x1+x2}{n1+n2}$ . In this question,  $\frac{37}{50}$  of the treatment group have been cured, and  $\frac{16}{50}$  of the sugar group have been cured. So  $p0 = \frac{37+16}{50+50} = 0.53$ . So the standard error  $= \sqrt{0.53(1-0.53)(\frac{1}{50}+\frac{1}{50})} = \boxed{0.0998, \text{ or C.}}$
- 26. This situation describes matched pairs design, or E.
- 27. The sugar pill would be a type of placebo or D.
- 28. By definition, the null hypothesis would be equal to 9 hours, and the alternative hypothesis would be less than 9 hours, or  $\boxed{D}$ , because the question states that the public isn't getting enough sleep.
- 29. This shows undercoverage bias, or A.
- 30. This is essentially a discrete probability distribution with values at  $x = 1.5, 2.5, 3.5, \ldots n + 0.5$  for positive integer n, and corresponding probabilities  $P(x) = \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \dots \frac{1}{2^n}$ . Then the expected value is  $\sum_{n=1}^{\infty} \frac{n+0.5}{2^n} = \sum_{n=1}^{\infty} \frac{n}{2^n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^n}$ . The latter sum is a geometric series with first term 1/2 and common ratio 1/2, so the sum is 1 (this also explains why the distribution is a valid pdf). The former sum is arithmetico-geometric, so it is a bit trickier. Let  $S = \sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \dots$  Then  $2S = 1 + \frac{2}{2} + \frac{3}{4} + \dots$  If we subtract the terms with corresponding denominators, we get  $S = 1 + \frac{1}{2} + \frac{1}{4} + \dots$  But this is just 1 plus the geometric series we evaluated to 1 earlier, so S = 2. Thus the final answer is  $2 + \frac{1}{2}(1) = \boxed{2.5$ , or B.}